Investigating Subtleties of the Multiplication Principle

Elise Lockwood
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Michigan State University Colloquium
January 23, 2018

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Zack Reed, Branwen Purdy, and John Caughman

Background and Motivation

- Counting problems are easy to state…
  but can be difficult to solve

Combinatorics Books Say Counting is Hard

- Martin’s (2001) first chapter is entitled “Counting is Hard”

  He points out that “there are few formulas and each problem seems to be different”
- Tucker (2002) says of his counting chapter, “we discuss counting problems for which no specific theory exists…it is the most challenging and most valuable chapter in this book”

Math Education Research Says Counting is Hard

- Eizenberg and Zaslavsky's (2004) findings "support the assertion that combinatorics is a complex topic – only 43 of the 108 initial solutions were correct"
  - English, 1993; Hadar & Hadass, 1981; Kavousian, 2006; Lockwood, 2014
- Broadly, my research goals are to learn everything I can about how to improve the teaching and learning of combinatorial enumeration

Outfits problem

- How many different shirt-pants-belt outfits can you make if you have 3 shirts, 4 pairs of pants, and 3 belts to choose from?

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3 ___ 4 ___ 3 ___

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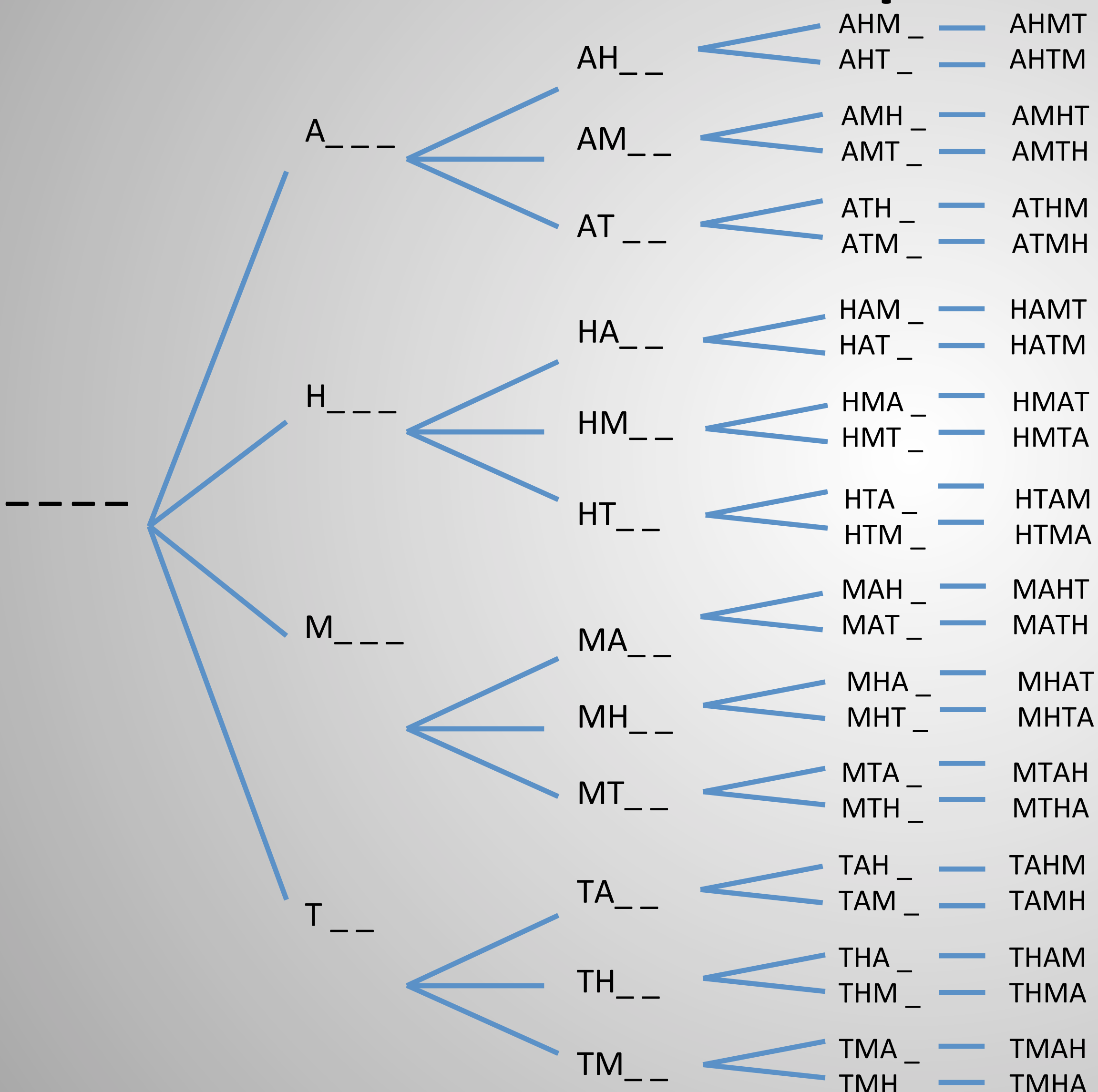
$\underline{\ 3\ } \times \underline{\ 4\ } \times \underline{\ 3\ } = 36$

MATH problem

- How many different ways are there to arrange the letters in the word MATH?

MATH problem



Set of Outcomes

AHMT	AMHT	ATHM
AHTM	AMTH	ATMH
HAMT	HMAT	HTAM
HATM	HMTA	HTMA
MAHT	MHAT	MTAH
MATH	MHTA	MTHA
TAHM	THAM	TMAH
TAMH	THMA	TMHA

$$4 \times 3 \times 2 \times 1 = 24$$

Question for You

- If you had to write a rule for when you should use multiplication to solve a counting problem, what would you write?

The Multiplication Principle

- "Fundamental Counting Principle"
- It underlies many of the counting formulas that students encounter

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$$

$${}_nP_r = \frac{n!}{(n-r)!} \qquad {}_nC_r = \frac{n!}{(n-r)!r!} = \binom{n}{r}$$

- The MP offers justification for why we get all of our desirable outcomes

The Multiplication Principle

Product Principle: *Let* $X_1, X_2, \ldots, X_k$ *be finite sets. Then, the number of* $k-$*tuples* $(x_1, x_2, \ldots x_k)$ *satisfying* $x_i \in X_i$ *is*

$$|X_1| \times |X_2| \times \ldots \times |X_k|.$$

The Product Rule: *Suppose that a procedure can be broken down into tasks. If there are* n_1 *ways to do the first task and* n_2 *ways to do the second task after the first task has been done, then there are* $n_1 n_2$ *ways to do the procedure.*

The Multiplication Principle: *Suppose a procedure can be broken into* m *successive (ordered) stages, with* r_1 *different outcomes in the first stage,* r_2 *different outcomes in the second stage, …, and* r_m *different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in the previous stages and if the composite outcomes are all distinct, then the total procedure has* $r_1 \times r_2 \times \ldots \times r_m$ *different composite outcomes.*

Motivation

- We started to observe some variety in MP statements

Two Related Studies

- A textbook analysis of statements of the MP
  - Capture the variety
  - See how statements of the MP are presented
- A reinvention study with a pair of undergraduate students
  - Interview two students over 8 hour-long sessions
  - Have them solve counting problems and then characterize when they multiply
  - Reinvent a statement of the MP

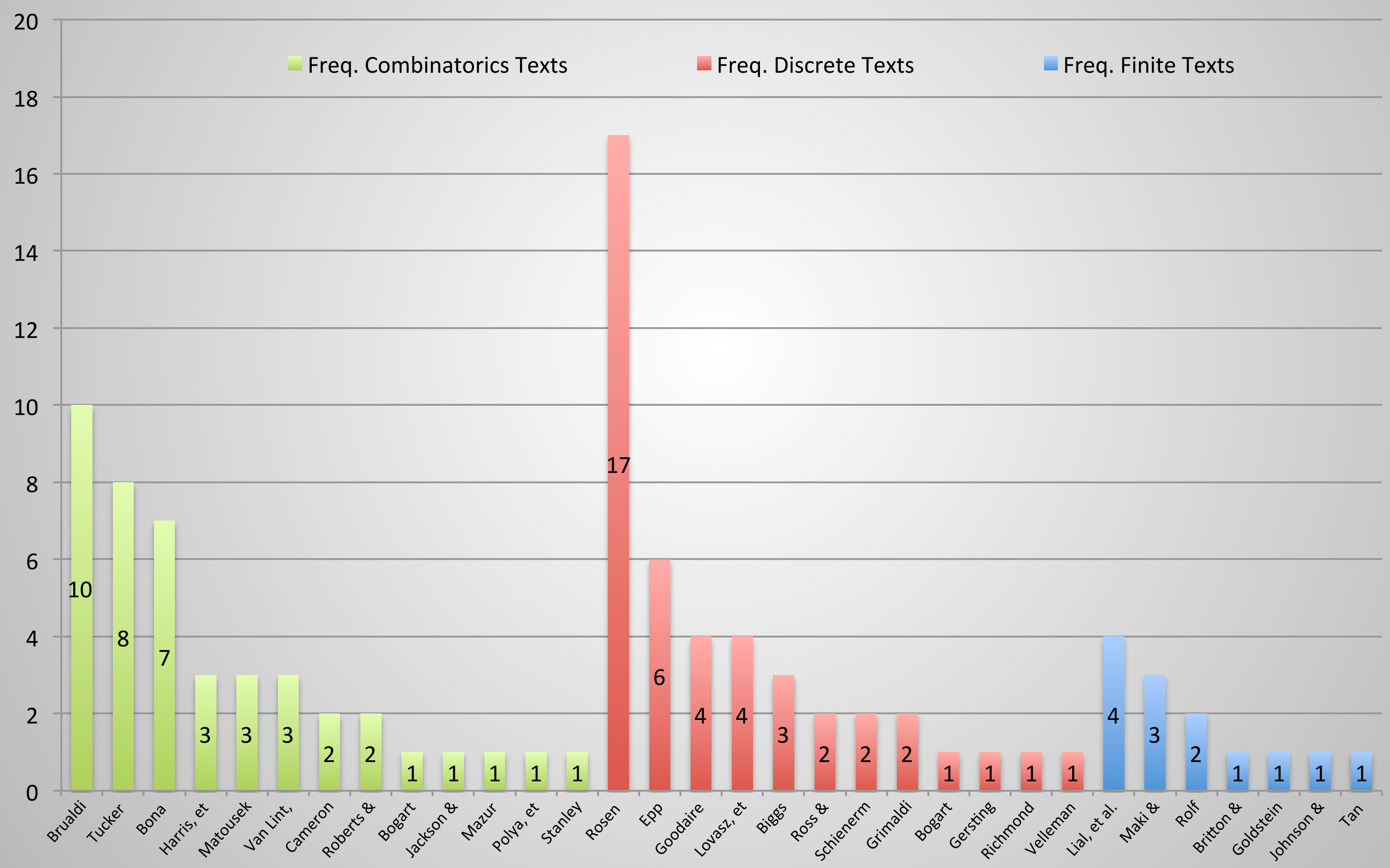
Research Questions

- How is the statement of the multiplication principle presented in postsecondary Combinatorics, Discrete Mathematics, and Finite Mathematics textbooks?
- What *mathematical issues* arise in comparing and contrasting different statements of the multiplication principle?
- How do *students reason about* key mathematical issues in the multiplication principle?

Study 1: Textbook Analysis

- We created a list of 70 universities from across the country
- We examined textbooks from these schools – 94 courses, yielding 32 textbooks

Frequencies of Textbooks



Textbook Analysis: Examining the Variety

- We created a list of 70 universities from across the country
- We examined textbooks from these schools – 94 courses, yielding 32 textbooks
- Another search among personal and university libraries yielded 32 more textbooks
- We had a final list of 64 textbooks, with 73 statements in total

Results of Textbook Analysis

Results of Textbook Analysis

- We want to highlight a key distinction that emerged from our analysis
  - 3 kinds of statements
    - Structural Statements
    - Operational Statements
    - Bridge Statements

3 Statement Types

Product Principle: *Let $X_1, X_2, \ldots, X_k$ be finite sets. Then, the number of $k-tuples$ $(x_1, x_2, \ldots x_k)$ satisfying $x_i \in X_i$ is*

STRUCTURAL

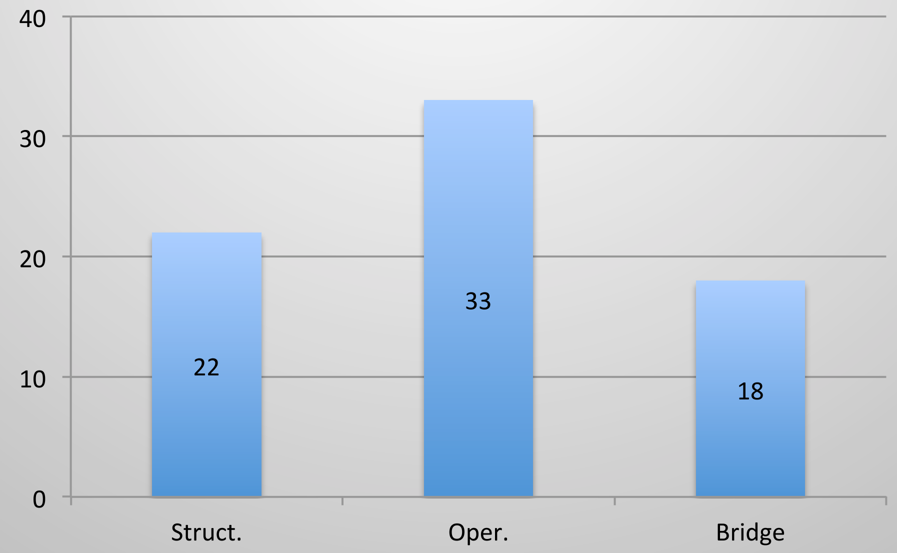
$$|X_1| \times |X_2| \times \ldots \times |X_k| .$$

The Product Rule: *Suppose that a procedure can be broken down into tasks. If tl* **OPERATIONAL** *e first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure.*

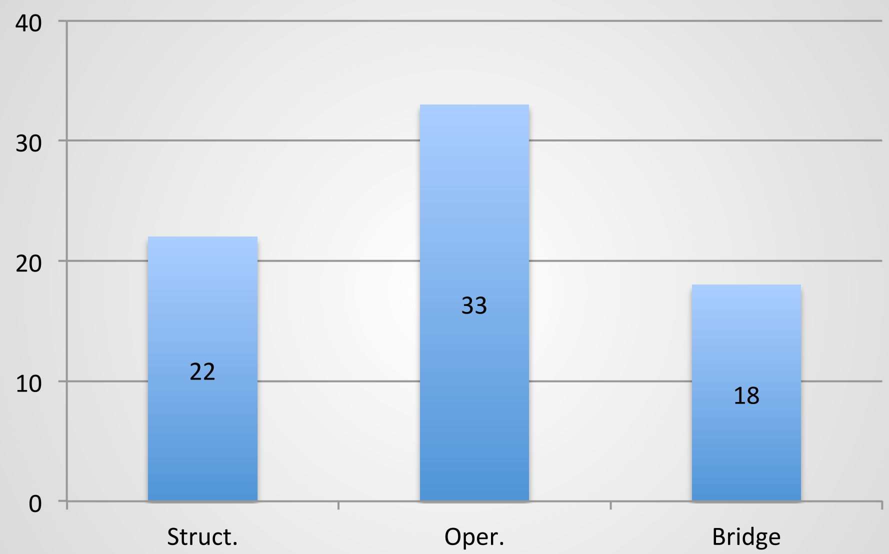
The Multiplication Principle: *Suppose a procedure can be broken into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, ..., and r_m different outcomes in the mth stage. If the number of outcon* **BRIDGE** *je is independent of the choices in the previous stages and if the composite outcomes are all distinct, then the total procedure has $r_1 \times r_2 \times \ldots \times r_m$ different composite outcomes.*

Summary of 3 Statement Types

Code	Criteria
Structural	The statement characterizes the MP as involving counting objects (such as lists or *k*-tuples)
Operational	The statement characterized the MP as determining the number of ways of completing a counting process
Bridge	The statement simultaneously characterizes the MP as counting outcomes and specifies a process by which those outcomes are counted



Why do we care?



- Mathematical implications of statement types

Three Mathematical Features

We identified 3 features central to many statements of the MP:

1. Require independence of # of options
2. Allow dependence of option sets
3. Require distinct composite outcomes

Feature 1:
Require independence of # of options

Outfits Problem: How many shirt-pants-belt outfits can be made from three different shirts, four different pairs of pants, and three different belts?

{S1, S2, S3} {P1, P2, P3, P4} {B1, B2, B3}

$\underline{\ 3\ } \times \underline{\ 4\ } \times \underline{\ 3\ } \quad = 36$

Feature 1:
Require independence of # of options

How many possible outcomes are there if I choose 2 different cards (with no replacement) from a standard 52-card deck, where the first is a face-card (J,Q,K) and the second is a heart?

A tempting answer is **12** x **13** =156 (# face cards x # hearts)

If I first choose one of the 9 **non-heart face cards**, there are 13 choices for the second card.
If I first choose one of the 3 **heart face cards**, there are only 12 choices for the second card.

So there are **9**x**13** + **3**x**12** = 153 possible outcomes.

Feature 1: Require independence of # of options

Product Principle: *Let $X_1, X_2, \ldots, X_k$ be finite sets. Then, the number of $k-$tuples $(x_1, x_2, \ldots x_k)$ satisfying $x_i \in X_i$ is*

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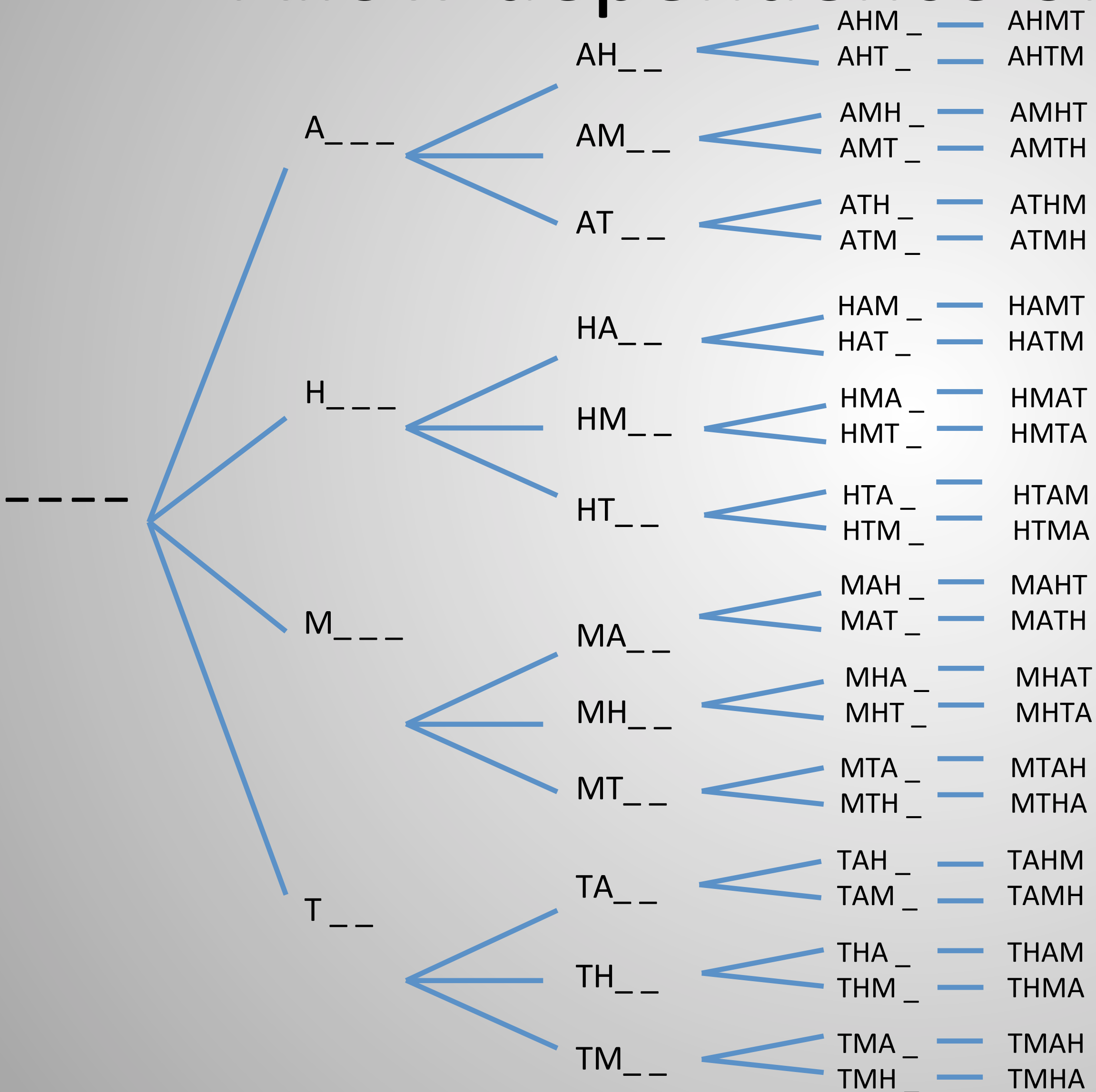
Feature 1: Require independence of # of options

- Of 51 operational or bridge statements
  - 11 attended to independence explicitly
  - 16 attended to independence implicitly
  - 24 statements did not attend to independence in the statement at all

Feature 2:
Allow dependence of option sets

How many different ways are there to arrange the letters in the word MATH?

Feature 2: Allow dependence of option sets



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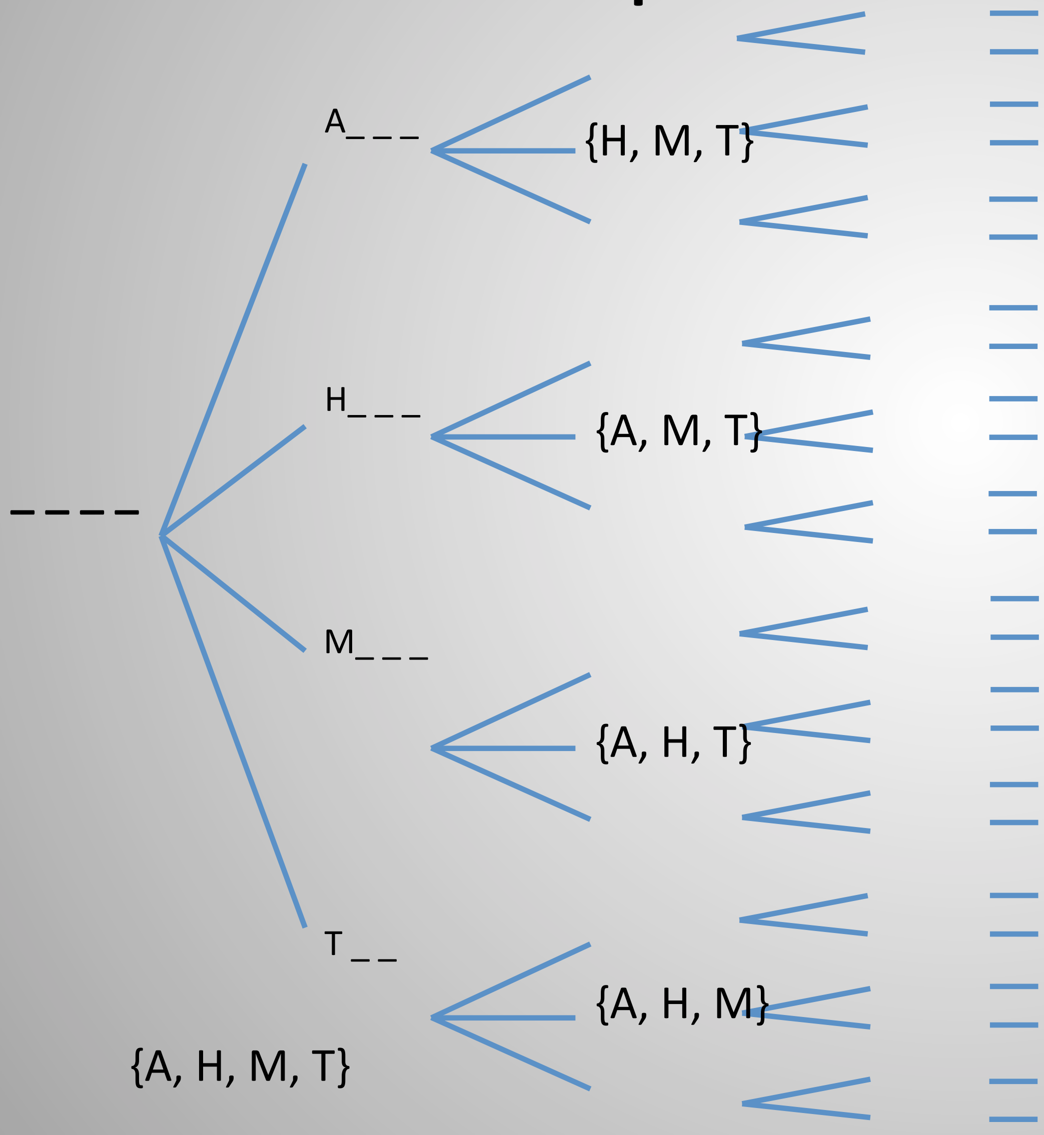

4 x 3 x 2 x 1 = 24

Feature 2:
Allow dependence of option sets

- The **cardinalities** are independent, even though the **sets** themselves may not be independent
- Feature 2 can be problematic for a purely structural statement
- This does not account for some simple situations when we'd like to multiply

Product Principle: *Let $X_1, X_2, \ldots, X_k$ be finite sets. Then, the number of $k-$tuples $(x_1, x_2, \ldots x_k)$ satisfying $x_i \in X_i$ is*

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Feature 3:
Require distinct composite outcomes

- *How many 3-letter sequences can be made using the letters a, b, c, d, e, f,*
  - *If the word* ***must contain*** *e, and* ***no repetition*** *of letters is allowed?*
  - 3 x 5 x 4

e ___ ___ ___

___ e ___ ___

___ ___ e

Feature 3:
Require distinct composite outcomes

- *How many 3-letter sequences can be made using the letters a, b, c, d, e, f,*
  - *If the word **must contain** e, and **no repetition** of letters is allowed?*
  - *If the word **must contain** e, and **repetition** of letters **is** allowed?*
  - 3 x 6 x 6

e ___ ___ ___

___ e ___ ___

___ ___ e ___

Feature 3:
Require distinct composite outcomes

- A purely operational statement can mislead on that last problem
  - 3 x 6 x 6 overcounts
  - Consider two ways of completing the process

e a e　　　e a e

  - The “eae” password is counted too many times

Feature 3: Require distinct composite outcomes

- A purely operational statement can mislead on that last problem
  - By an operational statement of the MP, there are 3 x 6 x 6 = 108 ways of completing the process
  - This is true, but this is not equal to the number of distinguishable desirable outcomes

The Product Rule: *Suppose that a procedure can be broken down into tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure.*

Feature 3: Require distinct composite outcomes

- A purely structural statement can also mislead on that last problem
  - 3 x 6 x 6, same 3 stages
  - Encode solutions as 3-tuples where
    - first coordinate is a number {1-3},
    - second coordinate is a letter {a-f}
    - third coordinate is a letter {a-f}

_e_ _a_ _e_   _e_ _a_ _e_

(1, a, e)   (3, e, a)

Feature 3: Require distinct composite outcomes

- A purely structural statement can also mislead on that last problem
  - By a structural statement, there are 3 x 6 x 6 = 108 different 3-tuples
  - This is true, but this is not equal to the number of distinguishable desirable outcomes

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Feature 3: Distinct composite outcomes

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Summarize Results of the Textbook Analysis

- We discovered a wide variety of MP statements
- There are three key mathematical features of the MP

Code	
Require independence of # of options	All 3 statement types (Structural, Operational, Bridge) can, and often do, address this issue.
Allow dependence of option sets	Structural statements do not naturally allow for this. Operational and Bridge statements can, and often do, address this issue.
Require distinct composite outcomes	Structural and Operational statements do not naturally address this issue. Bridge statements can, and often do, address this issue.

Conclusions and Implications of Study 1

If the MP statement you teach is:

- **structural**, your students may have issues with dependent choice sets or duplicate composite outcomes
- **operational**, your students may have issues with independence or duplicate composite outcomes
- **a bridge statement**, your students have no excuse!

Study 2: Student Understanding of the MP

- We worked with two calculus students in an 8-session teaching experiment
  - "A primary purpose for using teaching experiment methodology is for researchers to experience, firsthand, students' mathematical learning and reasoning" (Steffe & Thompson, 2000, p. 267)
  - Allows us to formulate and test hypotheses about students' reasoning over time
- Pat and Caleb (pseudonyms) were vector calculus students

Study 2: Student Understanding of the MP

- For the “teaching” in the teaching experiment, we used guided reinvention (Freudenthal, 1991)
  - Rather than give students statements to interpret, we give them tasks and experiences from which they can formalize mathematical ideas
  - We gave them tasks to perturb their thinking and iteratively refine their statements of the MP

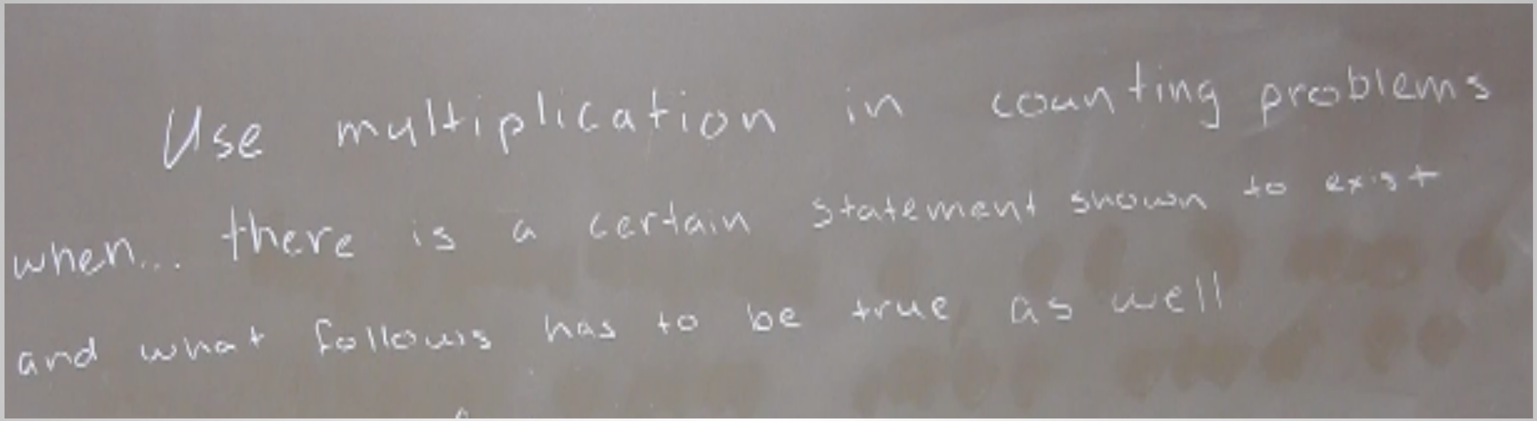
Study 2: Student Understanding of the MP

- Sessions 1-2 – Solving initial counting problems
  - Students gain experience using multiplication
  - They encounter initial mathematical issues
- Sessions 3-7 – Articulating and refining a statement of the MP
  - They progressively refine their statement
- Session 8 – Evaluating textbook statements

Initial MP Statement

- **If you had to write a rule for when you are going to use multiplication to solve counting problems, what would you write?**

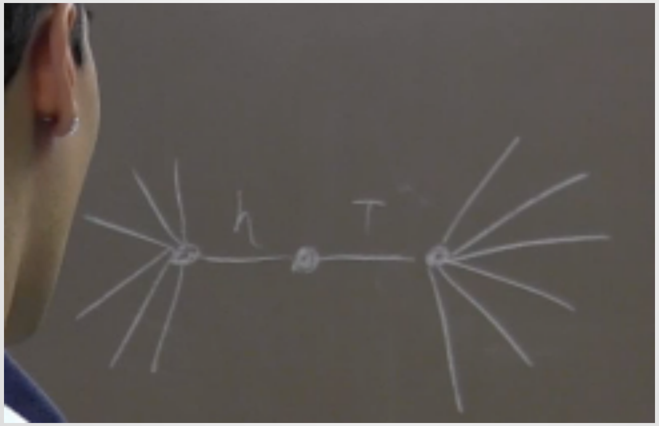
Initial MP Statement



Use multiplication in counting problems when… there is a certain statement shown to exist and what follows has to be true as well.

Coin, Die, Deck Task

How many ways are there to flip a coin, roll a die, and select a card from a standard deck?



Caleb*:* And off of those six possible options there will be 52 options for what cards you can pull from a deck.

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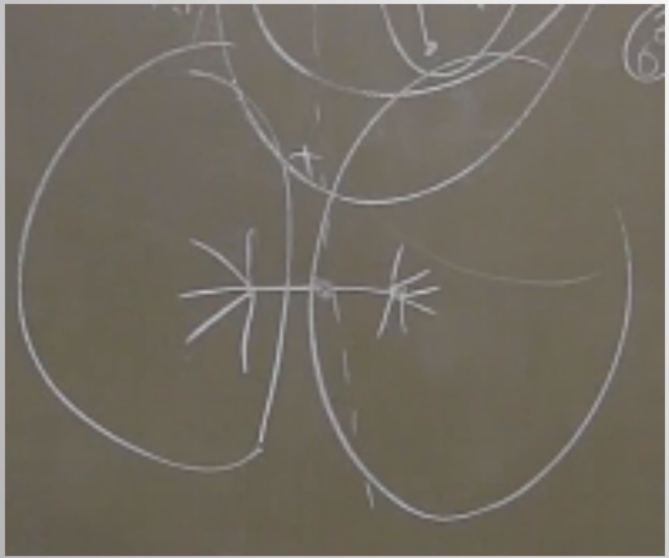

Caleb*:* So let's, we've definitely come to the conclusion that if their groups are equal we multiply.

Pat*:* If we're combining equal groups we're multiplying.

Refining a statement of the MP

Caleb: So for multiplication. How would we decide if they're equal or not?

Pat: Okay, um. If, for every possible selection, or for every possible outcome there's the same choices after that, for that.

Caleb: For every time?

Pat: For every possible outcome. Like for, like for the die. For every possible outcome of the die there is the same number of cards to select. And the same cards themselves. So like, I can, I can't figure out how to say the first part.

Caleb: Yeah that's exactly, that's how I feel.

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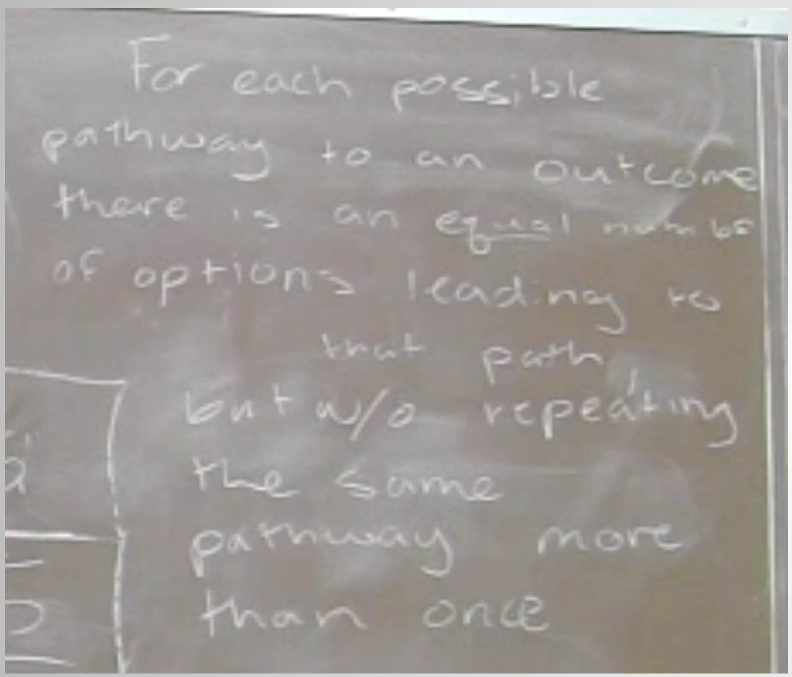
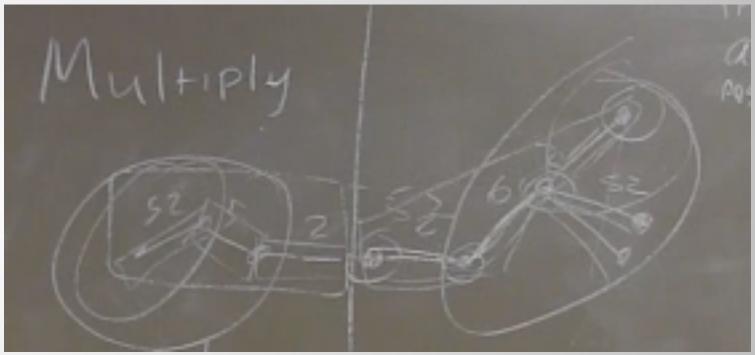
Statement 2

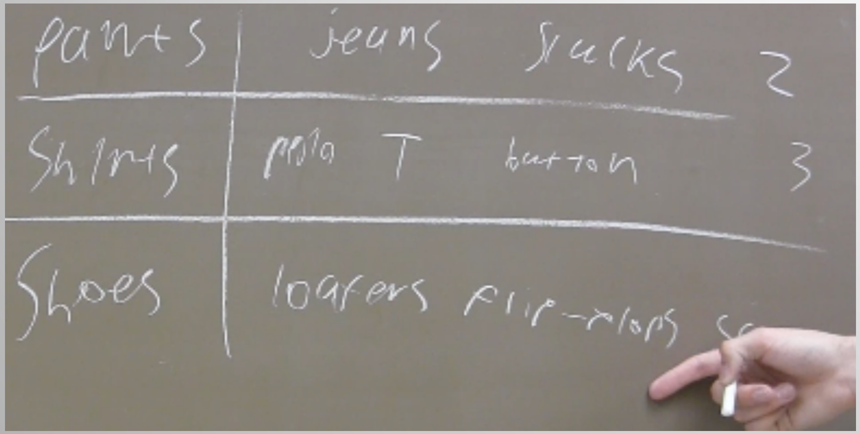


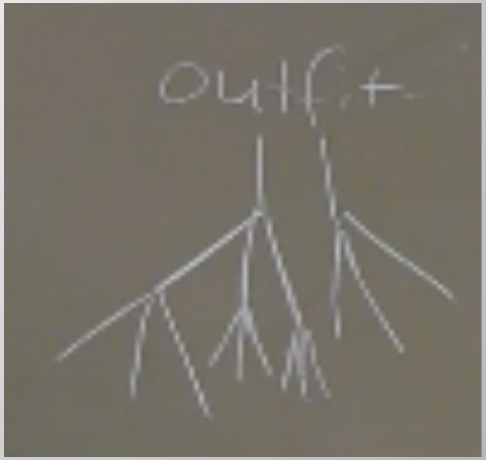


For each possible pathway to an outcome there is an equal number of options leading to that path but without repeating the same pathway more than once.

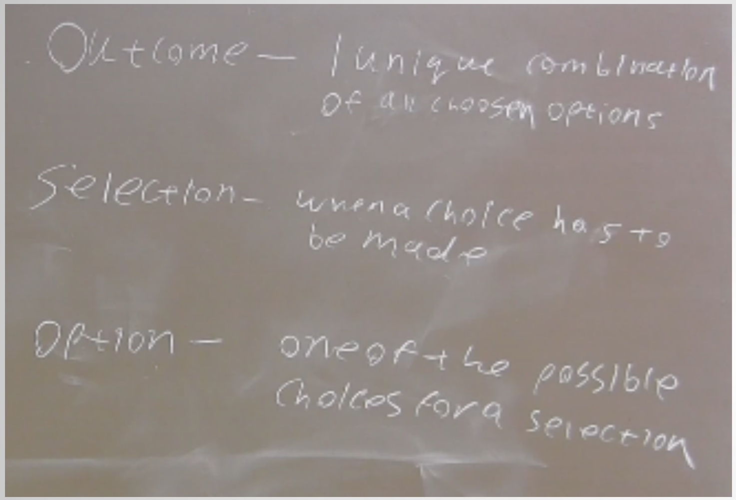
Push for More General Language

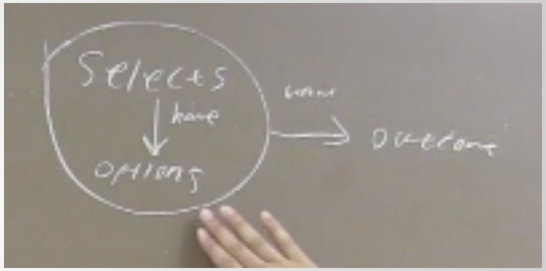
- We asked if they could articulate language that was more general than a "pathway"
- They spontaneously brought up a situation involving pants-shirts-shoes outfits in order to articulate the terms





Push for More General Language





- *A selection is when a choice has to be made*
- *An option is one of the possible choices for a selection*
- *An outcome is one combo of all chosen options*

Push for More General Language

- *"For every connected selection..."*

- ***Pat****:* Like the selection of pants, shirts, shoes makes sense for your outfit. But, like, if you said I select – after I get dressed, I'm gonna go eat breakfast, you know, it's not gonna make a whole lot of sense to say you know my outfit is going to decide if I have cereal.
- ***Caleb****:* At that point, it wouldn't be like outfit, it would be morning. Like a whole routine.

- *"For every selection towards a specific outcome…"*

Student Reasoning about Feature 1: Independence

How many possible outcomes are there if I choose 2 different cards (with no replacement) from a standard 52-card deck, where the first is a face-card (J,Q,K) and the second is a heart?

Student Reasoning about Feature 1: Independence

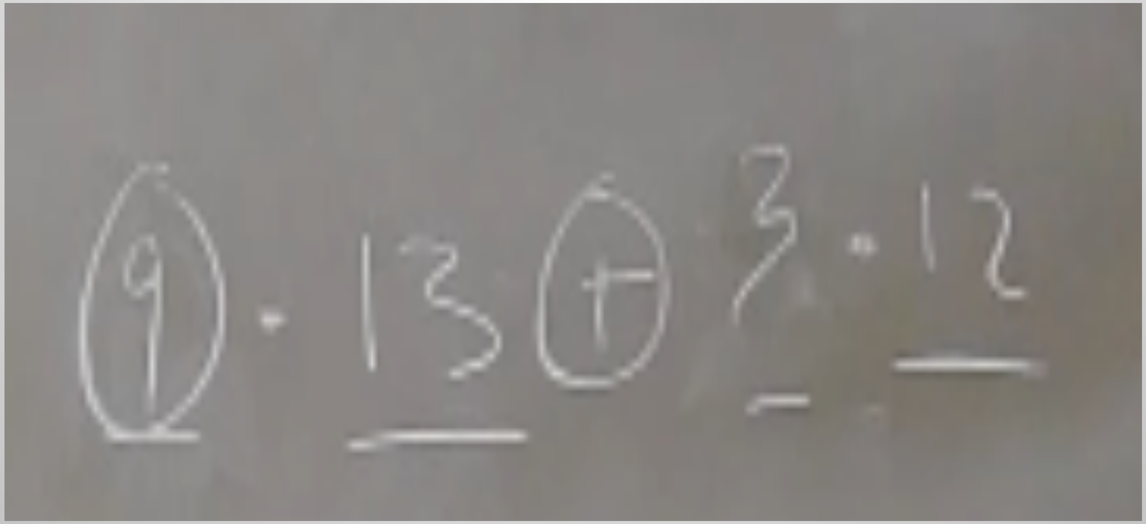
Pat: Yeah there are 12 options for your face card.

Caleb: Yeah, and then you'd multiply that by how many hearts there are which is 13, saying that your first one was a face card.

Pat: So there's 13 or 12 options depending on if the first one's a heart or not.

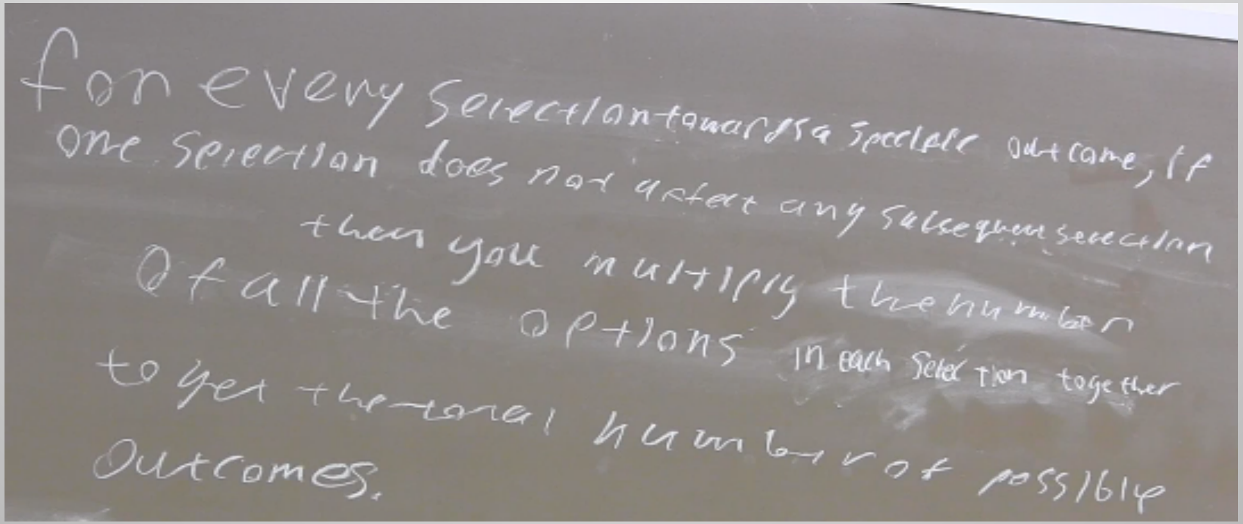
Student Reasoning about Feature 1: Independence

Caleb: Well I think we can just say that, oh gosh, this is hard. Oh, we could do if there is no heart and then we could do if there is a heart.



Statement 3b

For every selection towards a specific outcome, if one selection does not affect any subsequent selection then you multiply the number of all the options in each selection together to get the number of possible outcomes.



Student Reasoning about Feature 2: Dependence of Option Sets

How many 6-character license plates consisting of letters or numbers have no repeated character?

- 36 x 35 x 34 x 33 x 32 x 31

Student Reasoning about Feature 2: Dependence of Option Sets

Caleb: It'd be like 36 x 35 x 34 x 33 x 32 x 31.

Pat: Yeah and that wouldn't be multiplication as far as like –

Caleb: Yeah it would.

Pat: This goes to factorial multiplication, because towards our outcome – the "every selection here affects a subsequent selection." Whatever you select here restricts what could be here.

Student Reasoning about Feature 2: Dependence of Option Sets

Caleb: But it's still multiplication.

Pat: It's still multiplication but it's not the same as multiplication that we were thinking of. So, it has to change things now doesn't it?

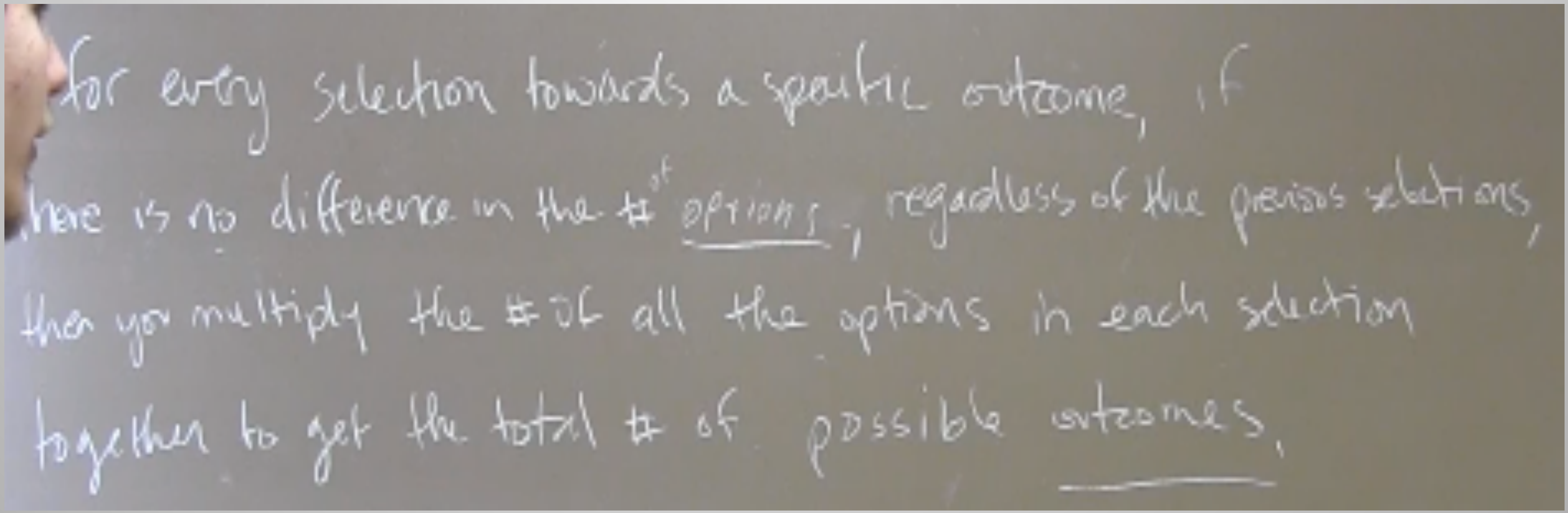
Caleb: That one definitely put a damper on our [statement].

Student Reasoning about Feature 2: Dependence of Option Sets

Pat: I'm just concerned about the idea that we're saying, that the selection is affecting the next selection, because technically in this case, every selection affects subsequent selections, but it still is multiplication.

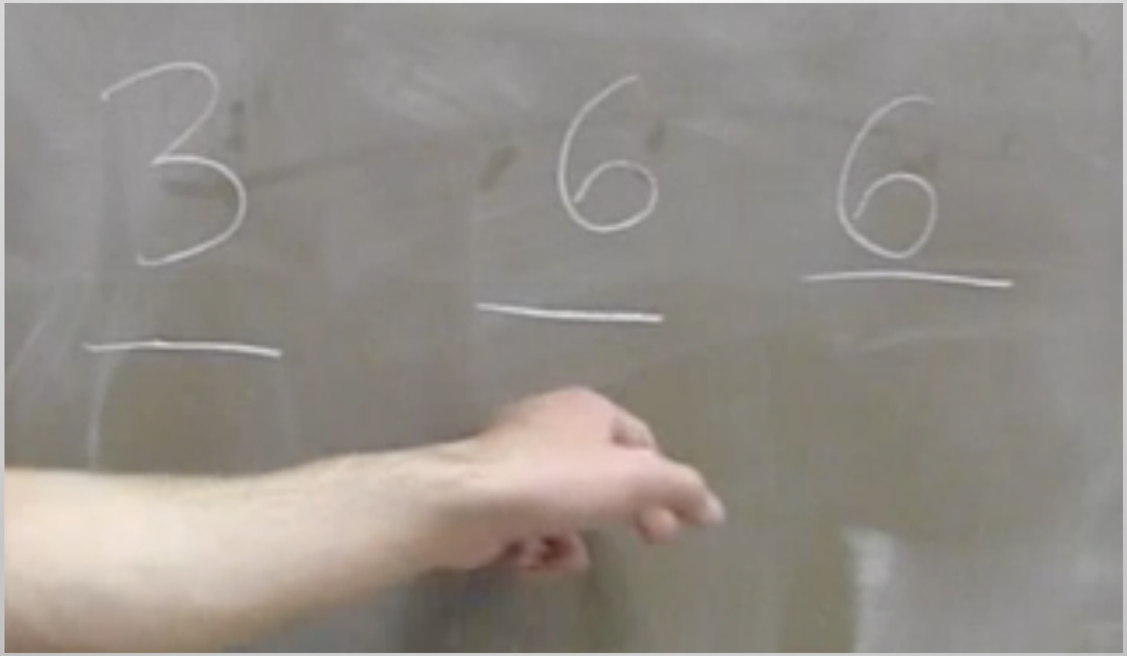
Statement 3c

If for every selection towards a specific outcome, if there is no difference in the ***number of options,*** *regardless of the previous selections, then you multiply the number of all the options in each selection together to get the total number of possible outcomes.*



Student Reasoning about Feature 3: Distinct Composite Outcomes

- We introduced a problem involving overcounting
  - The 3-letter sequences problem



Student Reasoning about Feature 3: Distinct Composite Outcomes

Caleb: **<u>So our problem here is over counting</u>**, and you can't just like put in a clause of like "don't over count."

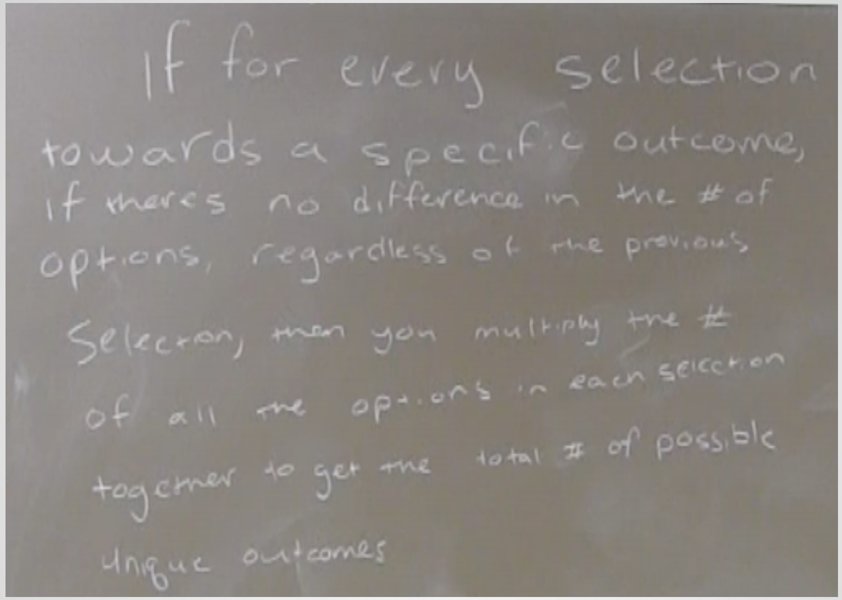
Pat: So how about we say specific unique outcome?

Caleb: Yeah.

Student Reasoning about Feature 3: Distinct Composite Outcomes

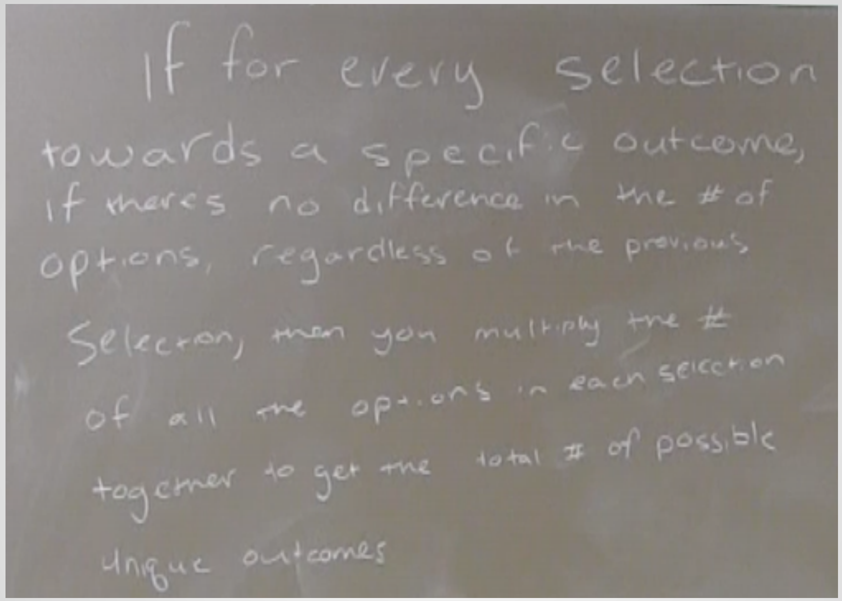
Pat: **So I feel like unique has to be added before the very last word.** 'Cause I feel like that at least grammatically takes care of this case. But it doesn't intuitively explain to you how to be sure of that.

Student Reasoning about Feature 3: Distinct Composite Outcomes



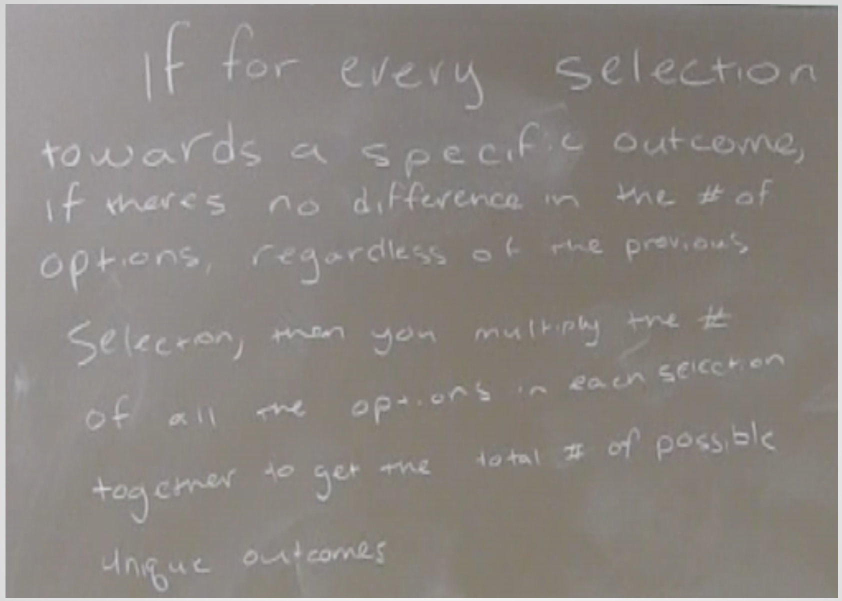
If for every selection towards a specific outcome, if there is no difference in the number of options, regardless of the previous selections, then you multiply the number of all the options in each selection together to get the total number of possible ***unique*** *outcomes.*

Final Statement



Pat: So it should count for all instances where nothing changes, it should count for factorial instances. And it should – at least with some amount of intuition and understanding, unique should make it so that you don't over count.

Final Statement



- This is a bridge statement
  - They are counting outcomes, but they describe a process of selections that generate those outcomes

Evaluating Textbook Statements

The Product Rule: *Suppose that a procedure can be broken down into tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure.*

Caleb: This kinda addresses none of the things we say, except for that you multiply them.

Evaluating Textbook Statements

Product Principle: *Let $X_1, X_2, \ldots, X_k$ be finite sets. Then, the number of $k-$tuples $(x_1, x_2, \ldots x_k)$ satisfying $x_i \in X_i$ is*

$$|X_1| \times |X_2| \times \ldots \times |X_k| .$$

Pat: Yeah I feel like, I think this is, **holds really strong intuitively for things that are completely separate**, like, uh there are no heads and tails on a die, if that makes sense.

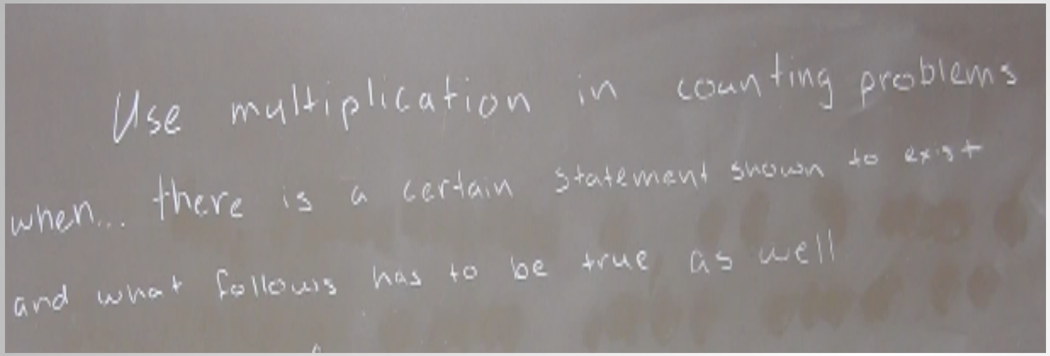
But **when you have it like where it is overlapping**, like I feel like this does work but it's a little hard, you have to do a little more intuitive thinking into it of like the idea that **you could overlap people and it doesn't matter, just as long as your cardinality stays the same**.

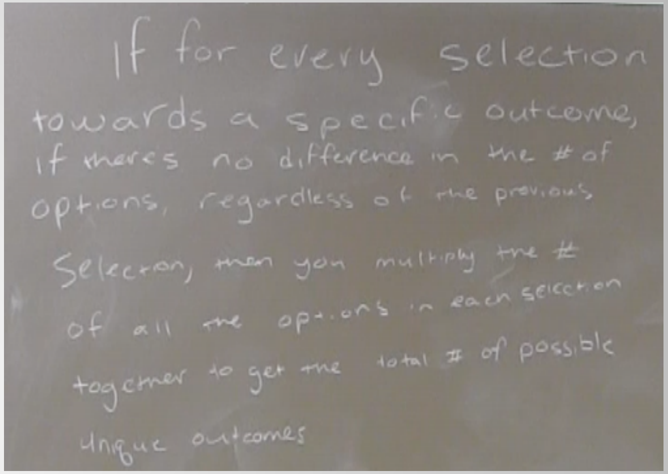
Evaluating Textbook Statements

The Multiplication Principle: *Suppose a procedure can be broken into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, ..., and r_m different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in the previous stages and if the composite outcomes are all distinct, then the total procedure has $r_1 \times r_2 \times \ldots \times r_m$ different composite outcomes.*

Caleb: That kinda gets back to ours. It addresses the independent choices and the unique outcomes. I think if we broke down each of ours we could basically reword them to be the same.

Summary of the Students' Reinvention





- The statements became more sophisticated throughout the teaching experiment
- By giving them problems that highlighted the three mathematical features we could help the students refine their statement

Summary of 3 Features

Feature	
Independent # of options	*All 3 statement types (Structural, Operational, Bridge) can, and often do, address this issue.* **Give problems that involve dependent stages in the process.**
Allow option sets to be dependent	*Structural statements do not naturally allow for this (a few ugly but notable exceptions exist.) Operational and Bridge statements can, and often do, address this issue.* **Give problems that involve permutations, or stages with decreasing numbers of options that do not simply involve Cartesian Products.**
Composite outcomes be distinct	*Structural and Operational statements do not naturally address this issue. Bridge statements can, and often do, address this issue.* **Give problems in which overcounting can occur.**

Conclusions and Implications

- The MP is a nuanced idea with subtle and important mathematical features
- If students do not grapple these subtleties, they may apply the MP without understanding potential issues that may arise

Conclusions and Implications

- Although these statements can be hard to interpret, through engaging with particular tasks our students became attuned to key mathematical features of the MP
  - Independence
    - Pairs of Cards
  - Dependence of option sets
    - Arrangements or permutations
  - Distinct composite outcomes
    - Three-letter sequences containing e

Conclusions and Implications

- Some advice for students of discrete mathematics
  - Think about what you are trying to count (outcomes) and how your counting process generates and structures those outcomes
  - Be thoughtful and careful about how you apply the MP when solving counting problems
  - Don't over count!

Thank You!!

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