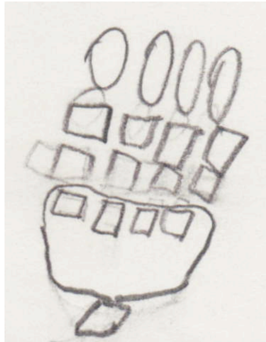

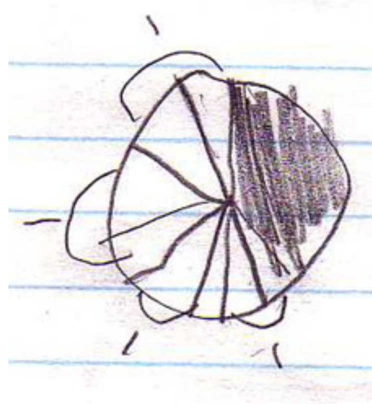

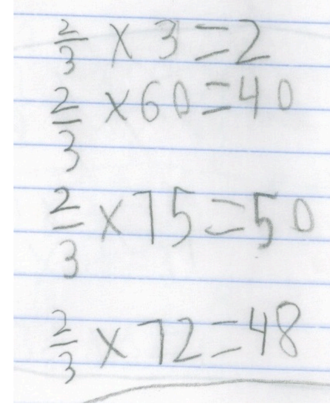

Seeing the Algebra in Children's Fraction Work

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Systemic, pervasive problem

- Many students do not understand the mathematics they learn

Systemic, pervasive problem

- Many students do not understand the mathematics they learn
- Watershed concepts
  - Base 10 and Place Value
  - Fractions
  - Equivalence & Equality
  - Fundamental Properties of Operations

What is understanding?

What is understanding?

- Connections

“For an idea to be understood, it must be related to other ideas”

-Carpenter and Lehrer, 1999

How is understanding developed?

- Mathematical practices

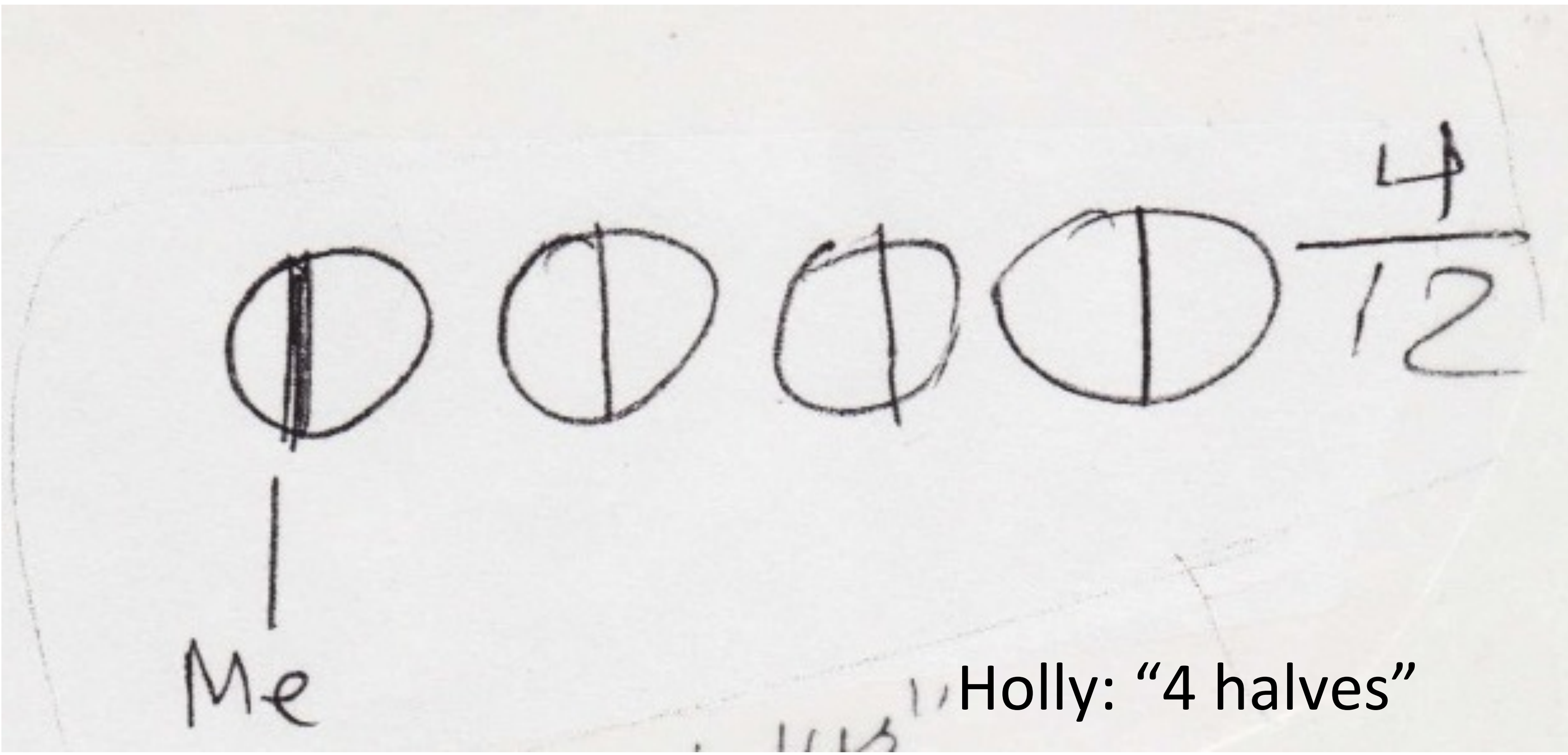
> “Scientific concepts are never developed without participation in specialized forms of practice” and “concepts are contingent on these practices”
>
> -Catley et al., 2005

Symptoms of the problem

A fifth grader's thinking

Jeremy is making cupcakes. He wants to put ½ cup of frosting on each cupcake. If he makes 4 cupcakes for his birthday party, how much frosting will he use to frost all of the cupcakes?

A fifth grader's thinking



Holly: "4 halves"

Holly’s thinking is not that unusual

- Difficulties with fractions widely acknowledged
  - Difficulties remembering and understanding procedures
  - Proficiency with procedures does not mean learners understand

More evidence of the problem

- TIMSS item for 8^{th} graders

Divide: $\frac{6}{55} \div \frac{3}{25} =$

Answer: ____________

- TIMSS item for 8th graders

$$\text{Divide: } \frac{6}{55} \div \frac{3}{25} =$$

Answer: ______________

- Answered correctly by 37% of U.S. students
  - International average 45%

- Another TIMSS item:

Laura had \$240. She spent $\frac{5}{8}$ of it. How much money did she have left?

Answer: ____________

- Another TIMSS item:

Laura had \$240. She spent $\frac{5}{8}$ of it. How much money did she have left?

Answer: ____________

- Answered correctly by 25% of U.S. students
  - International average 30%

- One more:

Which of these is the smallest number?

A. 0.625

B. 0.25

C. 0.375

D. 0.5

E. 0.125

- One more:

Which of these is the smallest number?

A. 0.625

B. 0.25

C. 0.375

D. 0.5

E. 0.125

- 51% of U.S. students answered correctly
  - International average 46%

Addressing the problem

A recent claim

- Lack of proficiency in fraction arithmetic impedes student progress in algebra

A recent claim

- Lack of proficiency in fraction arithmetic impedes student progress in algebra
- And an initial revision: *Lack of understanding the conceptual continuities between whole-number and fraction arithmetic impedes students' understanding of algebra*

Have you ever seen a student do this?

$$7a + 4a = 11a^2$$

Let's go back to arithmetic. What do these all have in common?

$$70 + 40$$

$$\tfrac{7}{5} + \tfrac{4}{5}$$

$$7a + 4a$$

How might children solve these? What do they have in common?

$$70 + 40 = 7 \times 10 + 4 \times 10 = (7 + 4) \times 10 = 110$$

$$\tfrac{7}{5} + \tfrac{4}{5}$$

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$$7a + 4a = (7 + 4)a = 11a$$

Characterizing these patterns

- Understand numbers in order to relate to operation
  - Base-10 structure of whole numbers
  - Fractions as multiplicative $\frac{a}{b} = a \times \frac{1}{b}$
- Use of same **fundamental property of operations and equality**
  - Distributive property of multiplication over addition

- Students who understand and use these relationships are using Relational Thinking
  - (Carpenter et al., 2003; Empson, et al., 2011; Empson & Levi, 2011)

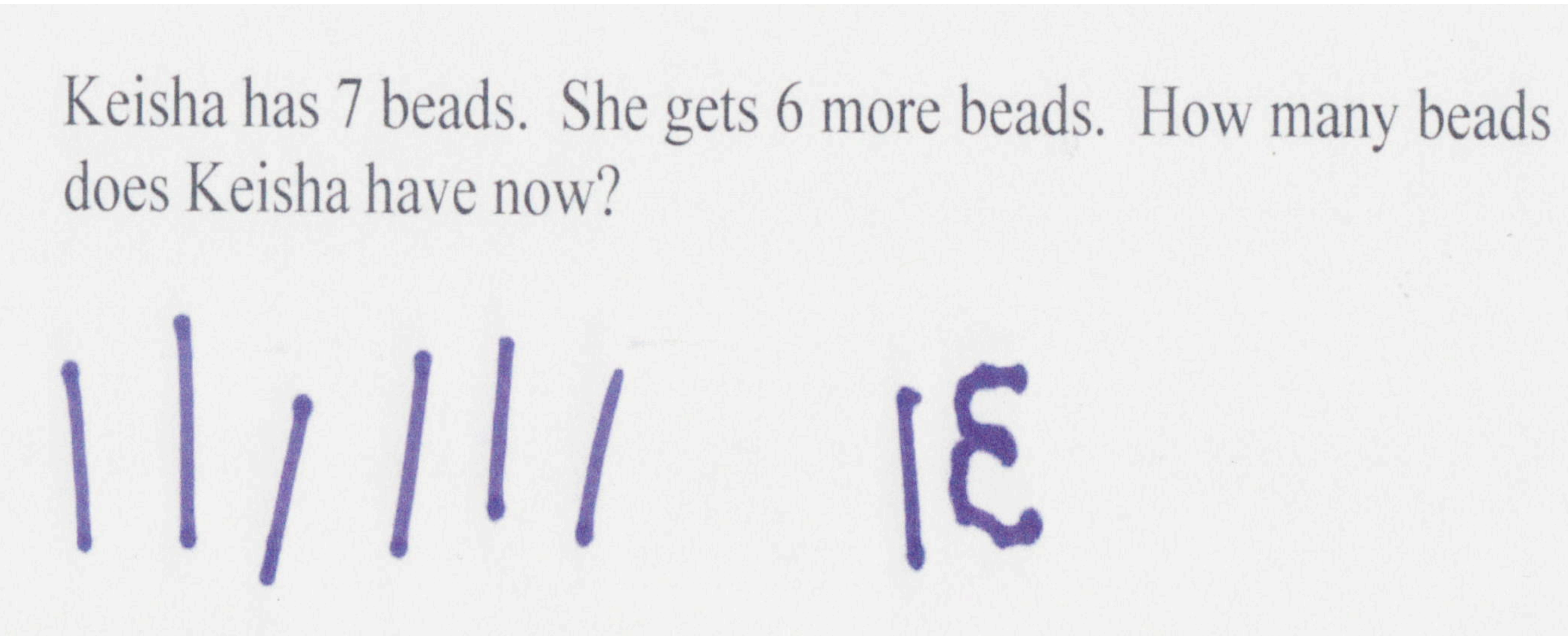
Relational Thinking

- Children draw upon their understanding of relations between operations and equality to reason about problems
  - Anticipatory view of the problem
  - Not as a series of steps to follow to get an answer
- Bridges arithmetic and algebra
  - Special set of relations: fundamental properties of arithmetic/algebra
  - Show up early
- Teacher's role is critical

Development of Relational Thinking

- Children use Relational Thinking spontaneously, naturally, to make sense of problems
- Begins early

Counting strategy



Keisha has 7 beads. She gets 6 more beads. How many beads does Keisha have now?

Relational Thinking strategy

Keisha has 7 beads. She gets 6 more beads. How many beads does Keisha have now?

7+3=10 7+6=13

10+3=13

Relationship used in Relational Thinking strategy

$$7 + 6 = 7 + (3 + 3) = (7 + 3) + 3$$

A problem given to first graders at end of unit of study

Tina and Tony painted pictures this afternoon. Tina used half a jar of blue paint for her picture. Tony used three fourths of the same sized jar of blue paint for his picture. How much blue paint did Tina and Tony use altogether for their paintings?

- 8 out of 17 students solved correctly
- “Three fourths has a half and a fourth in it. Put half with the half jar and that makes a whole jar. Then a fourth extra. So one whole jar and a fourth.”

Unpacking this thinking

- "Three fourths has a half and a fourth in it"

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$$

Unpacking this thinking

- "Three fourths has a half and a fourth in it"

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$$

- "Put half with the half jar and that makes a whole jar. Then a fourth extra."

$$\frac{1}{2} + \left(\frac{1}{2} + \frac{1}{4}\right) = \left(\frac{1}{2} + \frac{1}{2}\right) + \frac{1}{4}$$

Unpacking this thinking

- "Three fourths has a half and a fourth in it"

$$3/4 = 1/2 + 1/4$$

- "Put half with the half jar and that makes a whole jar. Then a fourth extra."

$$1/2 + (1/2 + 1/4) = \boxed{(1/2 + 1/2)} + 1/4$$

To figure $\frac{1}{2}+\frac{3}{4}$ the children used…

- Understanding of operation of addition
  - Involves combining like units
  - Associative property
    - A fundamental property of operations and equality

To figure $\frac{1}{2}+\frac{3}{4}$ the children used…

- Understanding of operation of addition
  - Involves combining like units
  - Associative property
    - A fundamental property of operations and equality
- Understanding of fractions
  - Decomposition of a fraction, equivalent fractions

To figure $\frac{1}{2} + \frac{3}{4}$ the children used…

- Understanding of operation of addition
  - Involves combining like units
  - Associative property
    - A fundamental property of operations and equality
- Understanding of fractions
  - Decomposition of a fraction, equivalent fractions
- Anticipatory thinking
  - Deciding what relationships can be used to simplify the problem. Decompose $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$

Claim

- With appropriate teacher support, the vast majority of children are capable of realizing the power of Relational Thinking

Supporting research

- 3-year longitudinal study of 1st, 2nd, and 3rd graders (Carpenter et al., 1998)
  - 82 children
  - 90% used Relational Thinking strategy for multidigit addition or subtraction at some point

- Year-long case study (Koehler, 2004)
  - All 2nd/3rd graders including six lowest performing students developed Relational Thinking strategies to make sense of whole-number multiplication
    - From not thinking relationally to Relational Thinking
      - Distributive property to partition facts
    - Also, marked improvement on district-mandated standardized test
- Other research

Use of Relational Thinking in multiplying and dividing fractions – Suggestive findings

Problem

Each cupcake takes $\frac{2}{3}$ cup of frosting. If Bety made 8 cups of frosting, how many cupcakes can she frost?

Problem

Each cupcake takes $2/3$ cup of frosting. If Bety made 8 cups of frosting, how many cupcakes can she frost?

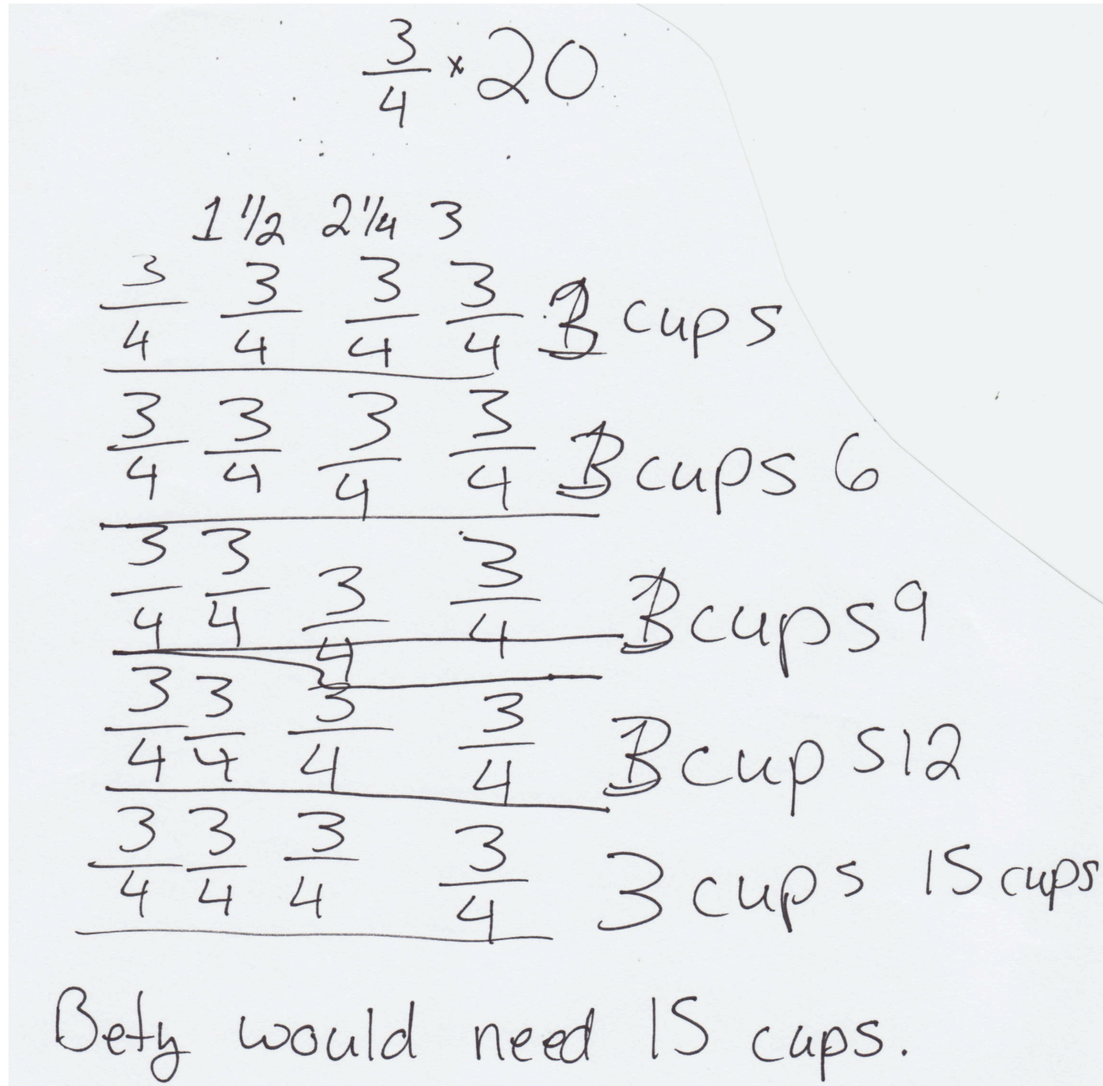
- Measurement Division (how many groups of?)
- $8 \div 2/3 = a$

Shana (5th grader)

- Each little cake takes ¾ of a cup of frosting. If Bety wants to make 20 little cakes for a party, how much frosting will she need?
  - Multiplication

Repeated addition strategy, with emergent grouping

- Shana's written work



Kylie (7th grader)

- Nina has $10\frac{1}{2}$ yards of fabric to make pillows. If each pillows takes $\frac{3}{8}$ of a yard of material, how many pillows can Nina make before she runs out of fabric?
  - Measurement division

Relational Thinking strategy

- Kylie's goal
  - How many $\frac{3}{8}$ does it take to make $10\frac{1}{2}$?

$$j \times \frac{3}{8} = 10\frac{1}{2}$$

What do you think Kylie's reasoning is?
What fundamental properties do you see?

$$8 \times \frac{3}{8} = 3$$

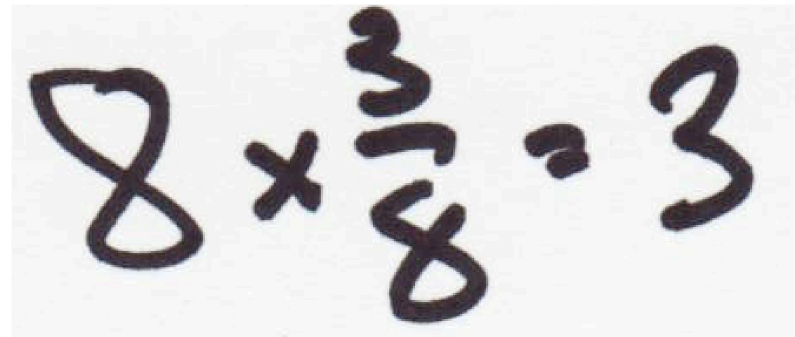
$$24 \times \frac{3}{8} = 9$$

$$28 \times \frac{3}{8} = 10\frac{1}{2}$$

- “8 groups of $\frac{1}{8}$ is 1, so 8 groups of $\frac{3}{8}$ would be 3”
- Always true?

$$8 \times \frac{3}{8} = 3$$

- "8 groups of $\frac{1}{8}$ is 1, so 8 groups of $\frac{3}{8}$ would be 3"
  - Always true?
- As understanding grows, learners identify and use generalized relationships such as this one



$$m \times {}^{n}\!/\!{}_{m} = n$$

Other relationships

$$8 \times {}^{3}\!/_{8} = 3$$

$$3 \times (8 \times {}^{3}\!/_{8}) = 3 \times 3$$

$$(3 \times 8) \times {}^{3}\!/_{8} = 9$$

- Multiplication Property of Equality
- Associative Property of Multiplication

- Now that you know $8 \times {}^{3}\!/_{8} = 3$, how could you use it to find the value for *b* that makes this equation true?

$$9 \times {}^{3}\!/_{8} = b$$

  - What fundamental properties did you use in your reasoning?

- Did you get $3\frac{3}{8}$?
- Did you use the distributive property?

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- Did you use the distributive property?

$$
\begin{aligned}
9 \times {}^{3}\!/_{8} &= (8+1) \times {}^{3}\!/_{8} \\
&= (8 \times {}^{3}\!/_{8}) + (1 \times {}^{3}\!/_{8}) \\
&= 3 + {}^{3}\!/_{8} \\
&= 3\,{}^{3}\!/_{8}
\end{aligned}
$$

- Did you get $3\,{}^{3}\!/_{8}$?
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&= 3 + {}^{3}\!/_{8} \\
&= 3\,{}^{3}\!/_{8}
\end{aligned}
$$

- Of course, there are other ways to solve this problem

Deon

$$63\frac{3}{4}\times\frac{2}{3}$$

Deon

$$63\frac{3}{4}\times\frac{2}{3}$$

$$63\,\frac{3}{4}\times\frac{2}{3}=\left(63\times\frac{1}{3}\times 2\right)+\left(\frac{3}{4}\times\frac{1}{3}\times 2\right)$$

$$21\times 2+\frac{1}{4}\times 2$$

$$42+\frac{1}{2}=42\,\frac{1}{2}$$

Standard procedure can be less efficient than Relational Thinking

$$63\frac{3}{4} \times \frac{2}{3} = \frac{255}{\not{4}_{2}} \times \frac{\not{2}^{1}}{3} = \frac{255}{6} = 42\frac{3}{6} = 42\frac{1}{2}$$

$$\begin{array}{r} {}^{1} \\ 63 \\ \times\ 4 \\ \hline 252 \end{array}$$

$$\begin{array}{r} 42\ R\ 3 \\ 6\overline{)255} \\ 24 \\ \hline 15 \\ 12 \\ \hline 3 \end{array}$$

As understanding grows, learners become more flexible in their use of relationships

- Solve some of these equations using Relational Thinking

$$k \times 3 = \frac{1}{2}$$

$$3 \times \frac{1}{4} = n \times \frac{1}{12}$$

$$m \div \frac{1}{8} = 7$$

$$a \times .1 + b = 3.4$$

Learning fractions with understanding

- Equal Sharing as a model
- Development of relations
  - Quantities
  - Use of Fundamental Properties to reason about operations on fractions
- Purposeful view of problem solving
  - Cultivate habit of making sense by drawing on mathematical relations—connections
  - Anticipatory thinking

Why Equal Sharing?

- Build on children's informal knowledge of sharing and partitioning
- Many different entry points to solve
- Develop understanding of what a fraction is
- Extend children's knowledge of division

Solve as a fourth-grader might

- What type of division?
- What kinds of strategies?
- What is the mathematics that comes up in children's strategies?

6 gators want to share 10 pies equally. How much pie can each gator have?

- Children understand fractions in terms of relationships

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$$1 \div 4 = \tfrac{1}{4} \quad \textit{and} \quad 4 \times \tfrac{1}{4} = 1$$

- Children understand fractions in terms of relationships

$1 \div 4 = \frac{1}{4}$ *and* $4 \times \frac{1}{4} = 1$

$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$

- Children understand fractions in terms of relationships

$1 \div 4 = \frac{1}{4}$ *and* $4 \times \frac{1}{4} = 1$

$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$

$3 \times \frac{1}{4} = \frac{3}{4}$

…..

One View

- Proficiency in fractions is foundational to algebra

$$7/5 + 4/5 = 11/5$$

$$9 \times 3/8 = 9/1 \times 3/8 = 27/8 = 3\,3/8$$

Another view

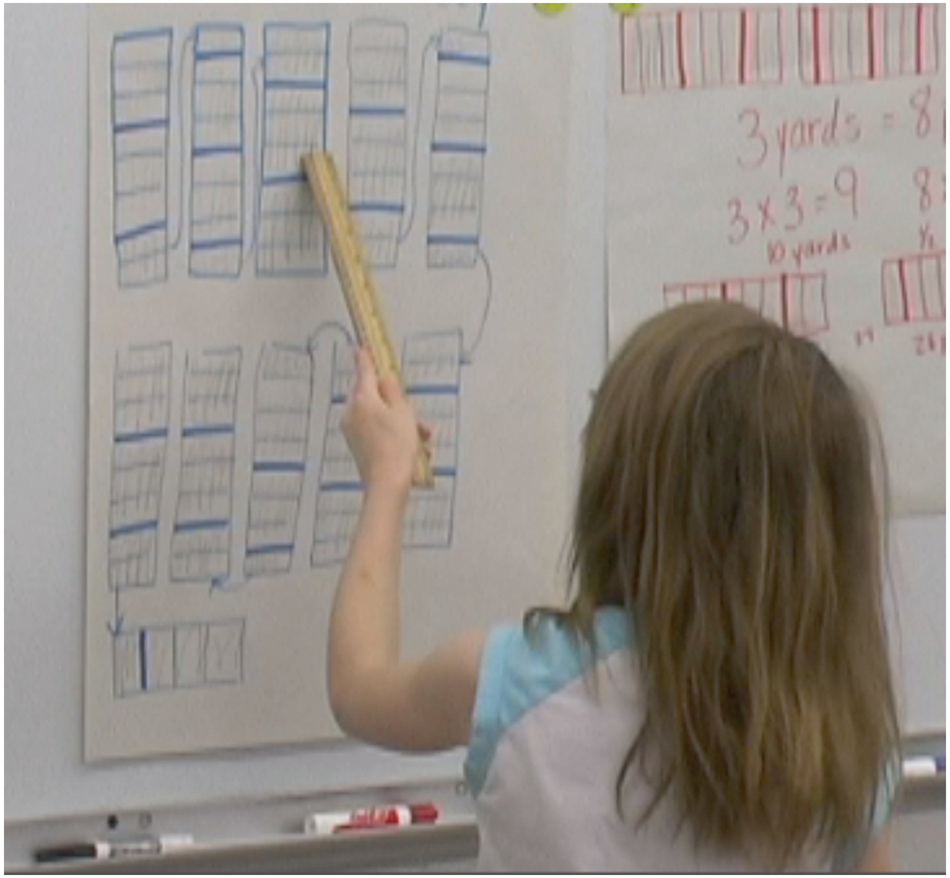
- *Lack of the development of Relational Thinking in elementary school impedes students' progress and understanding of algebra*
  - *Involving whole numbers and fractions*

Connections

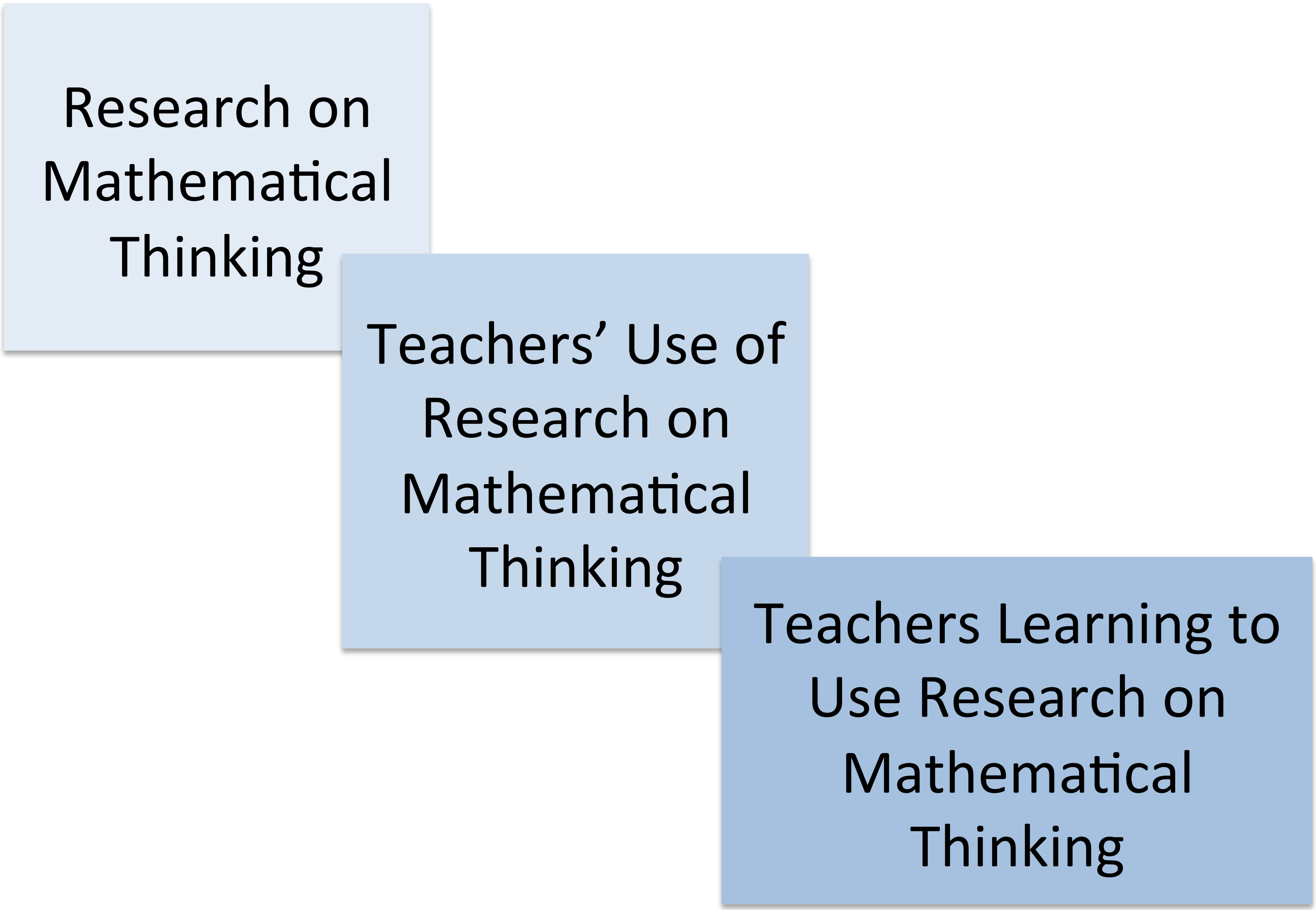
- But not all connections of equal value in understanding
- Relational Thinking specifies connections critical to learning whole-number and fraction arithmetic with understanding

- Relational Thinking across the arithmetic curriculum may be most critical precursor to learning algebra with understanding
  - Small set of Fundamental Properties govern children's reasoning about whole numbers and fractions
  - Learners use these properties naturally and purposefully in their strategies
    - If encouraged to use own thinking about how to solve

Where next?

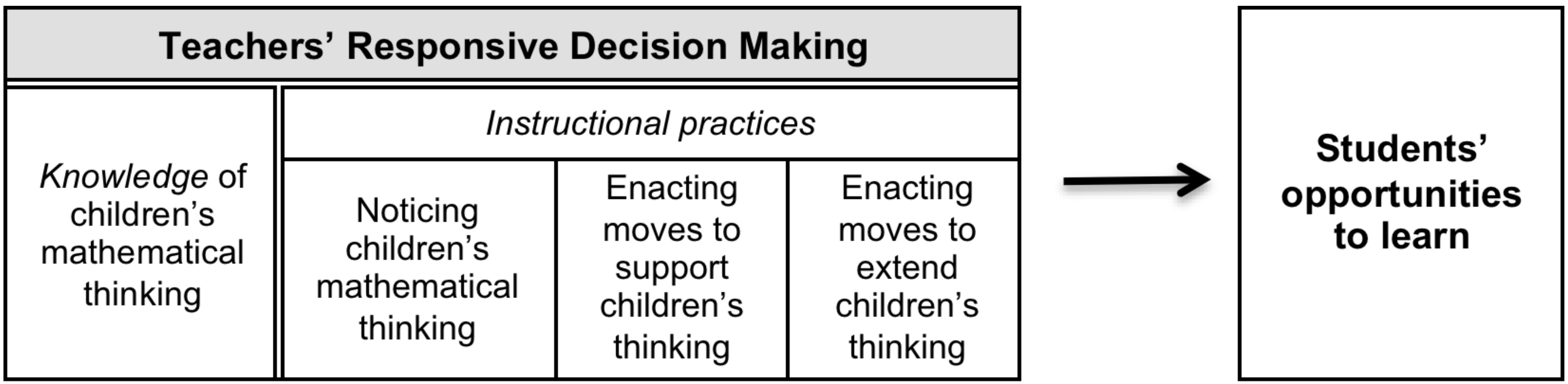


- Research-based knowledge of children’s thinking has proliferated
- Limited research on
  - How teachers use this knowledge
  - How to help teachers learn to use this knowledge
  - What’s possible in terms of children’s thinking when teachers use this knowledge


Research on Mathematical Thinking
Teachers' Use of Research on Mathematical Thinking
Teachers Learning to Use Research on Mathematical Thinking

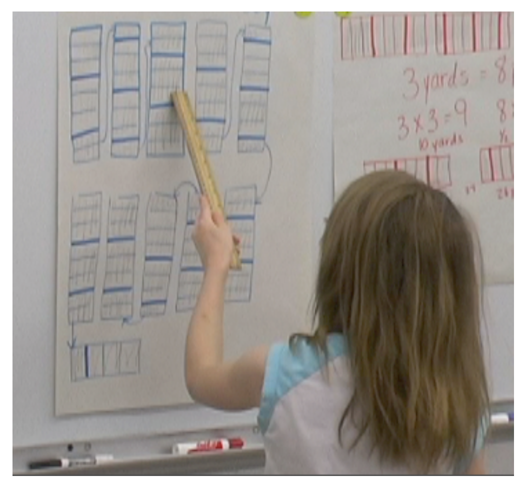
One take on teachers' use of research on mathematical thinking

Responsive decision making

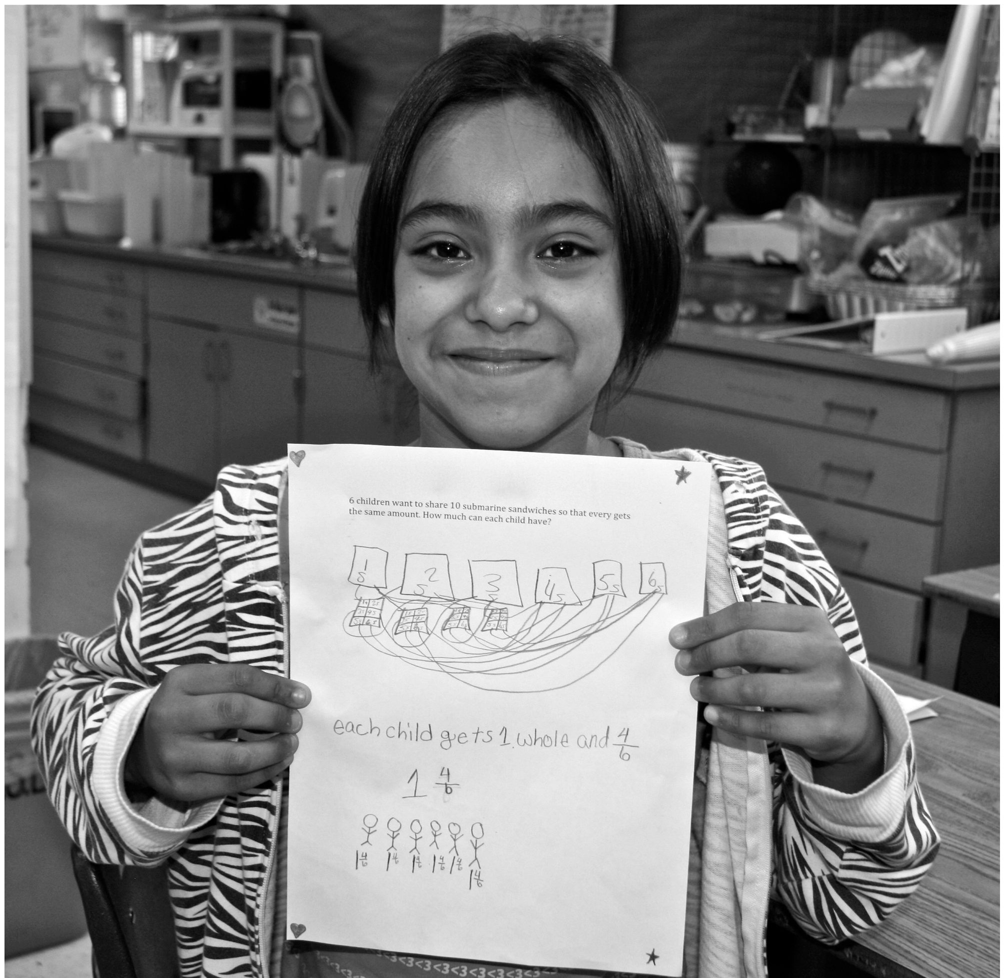


Role of the teacher

- Choose and adapt problems
  - Responsive to children's thinking
  - Address substantive mathematics
- Notate children's solutions and explanations
  - Problem situation
  - Strategy
  - Structure
- Sequence problems
- Listen, interpret, adapt



Thank you!


6 children want to share 10 submarine sandwiches so that every gets the same amount. How much can each child have?
each child gets 1 whole and 4/6
1 4/6